ORTHOGONALITY OF COSETS RELATIVE TO IRREDUCIBLE CHARACTERS OF FINITE GROUPS

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Abstract. Studied is an assumption on a group that ensures that no matter how the group is embedded in a symmetric group, the corresponding symmetrized tensor space has an orthogonal basis of standard (decomposable) symmetrized tensors.

0. Introduction

Let V be a complex inner product space and let G be a subgroup of the symmetric group S_n for some n. Corresponding to an irreducible character of G is a *symmetrizer*, a certain endomorphism of the n-fold tensor space $V^{\otimes n}$. The image under a symmetrizer of a standard basis vector of $V^{\otimes n}$ is called a *standard* (or *decomposable*) *symmetrized tensor*.

We seek conditions under which $V^{\otimes n}$ will have an orthogonal basis consisting entirely of standard symmetrized tensors (such a basis being called an *o-basis* for brevity). The problem of finding such conditions was first considered by Wang and Gong in [WG] where it was shown that if G is the dihedral group of order eight (viewed naturally as a subgroup of S_4), then $V^{\otimes 4}$ has an o-basis. In subsequent work [HT], Tam and the author showed that more generally if G is a dihedral group of order a power of two, then the corresponding tensor space has on o-basis. Moreover, it was noted there that this fact is independent of the particular embedding of the dihedral group inside the symmetric group. So, for instance, an o-basis exists for $V^{\otimes n}$ where n is the order of the dihedral group (still assumed to be a power of two) and the embedding is the one given by Cayley's Theorem.

Motivated by this example, we give in this paper conditions on a finite group ensuring that, regardless of how it is embedded in a symmetric group, the corresponding tensor space will have an o-basis. We call a group satisfying this condition an *o-basis group*.

In Section 1, we state the definition of an o-basis group and establish some properties. In Section 2, we review more carefully the notion of an o-basis of a tensor space and then give connections between this notion and that of an o-basis group. Finally, we show in Section 3 that the class of o-basis groups contains some interesting groups–the extra special p-groups (p, prime), for example.

1. Main definition and some properties

Let G be a finite group and let H be a subgroup of G. Denote by G/H the set of (left) cosets of H in G. The natural left action of G on the set G/H extends linearly to the complex vector space having this set as basis, which we denote by $\mathbf{C}(G/H)$.

Let $\operatorname{Irr}(G)$ denote the set of irreducible characters of G. Let $\chi \in \operatorname{Irr}(G)$. Define a form B_H^χ on $\mathbf{C}(G/H)$ by putting

$$B_H^\chi(aH, bH) = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(a^{-1}bh)$$

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(where e is the identity element of G) and extending linearly in the first component and antilinearly in the second.

1.1 Proposition. *B_H^χ is a well-defined G-invariant Hermitian form.*

Proof. First note that since χ is conjugation invariant [I, (2.3), p. 14], we have $\chi(ga) = \chi(g^{-1}gag) = \chi(ag)$ for all $a, g \in G$.

Suppose $a_1H = aH$ and $b_1H = bH$ so that $a_1 = ax$ and $b_1 = by$ for some $x, y \in H$. Then for each $h \in H$,

$$\chi(a_1^{-1}b_1h) = \chi(x^{-1}a^{-1}byh) = \chi(a^{-1}byhx^{-1}).$$

As h ranges through H, yhx^{-1} also ranges through H, so $B_H^\chi(a_1H, b_1H) = B_H^\chi(aH, bH)$ and B_H^χ is well-defined.

To say that B_H^χ is G-invariant is to say that $B_H^\chi(caH, cbH) = B_H^\chi(aH, bH)$ for all $a, b, c \in G$ and this is clear.

Finally,

$$\begin{aligned} B_H^\chi(bH, aH) &= \frac{\chi(e)}{|H|}\sum_{h\in H}\chi(b^{-1}ah) = \frac{\chi(e)}{|H|}\sum_{h\in H}\overline{\chi(h^{-1}a^{-1}b)} \\ &= \frac{\chi(e)}{|H|}\sum_{h\in H}\overline{\chi(a^{-1}bh^{-1})} = \overline{B_H^\chi(aH, bH)} \end{aligned}$$

($a, b \in G$), where we have used that $\overline{\chi(g)} = \chi(g^{-1})$ for $g \in G$ [I, (2.15), p. 20]. Therefore, B_H^χ is Hermitian. □

Put $\mathcal{C}_H^\chi := \mathbf{C}(G/H)/\ker B_H^\chi$, where $\ker B_H^\chi := \{x \in \mathbf{C}(G/H) : B_H^\chi(x, y) = 0 \text{ for all } y \in \mathbf{C}(G/H)\}$. Then B_H^χ induces a well-defined form $\bar{B}_H^\chi$ on $\mathcal{C}_H^\chi$ given by $\bar{B}_H^\chi(\overline{x}, \overline{y}) = B_H^\chi(x, y)$ ($x, y \in \mathbf{C}(G/H)$), where here and below we use $\overline{x}$ to denote the coset $x + \ker B_H^\chi$ (context should keep any confusion from arising over this notation and the usual notation for complex conjugate which we also use). By 1.1, $\ker B_H^\chi$ is closed under the action of G and so we have a well-defined action of G on $\mathcal{C}_H^\chi$. Clearly, $\bar{B}_H^\chi$ is G-invariant.

For characters ψ and φ of G, one defines $(\psi, \varphi)_H = \frac{1}{|H|}\sum_{h\in H}\psi(h)\overline{\varphi(h)}$ [I, (2.16), p. 20]. We denote the principal character of G by 1 (so $1(g) = 1$ for all $g \in G$).

1.2 Theorem.

(1) $\dim_{\mathbf{C}} \mathcal{C}_H^\chi = \chi(e)(\chi, 1)_H$.
(2) *The form $\bar{B}_H^\chi$ is positive definite.*

Proof. Let $a_1H, \dots, a_nH$ be the distinct left cosets of H in G. Then $\{a_iH : 1 \le i \le n\}$ is a basis for $\mathbf{C}(G/H)$. Let A be the $n \times n$-matrix with (i, j)-entry $B_H^\chi(a_iH, a_jH)/|G : H|$, where $|G : H|$ is the index of H in G. (So A is $|G : H|^{-1}$ times the matrix of the form B_H^χ relative to the above basis.) We claim that $A^2 = A$. The (i, j)-entry of A^2 is

$$\begin{aligned} |G:H|^{-2}\sum_{k=1}^n B_H^\chi(a_iH, a_kH)B_h^\chi(a_kH, a_jH) &= \frac{\chi(e)^2}{|G|^2}\sum_{k=1}^n\left(\sum_{h\in H}\chi(a_i^{-1}a_kh)\right)\left(\sum_{l\in H}\chi(a_k^{-1}a_jl)\right) \\ &= \frac{\chi(e)^2}{|G|^2}\sum_{l\in H}\sum_{\substack{h\in H\\1\le k\le n}}\chi(a_kha_i^{-1})\chi(a_jlh^{-1}a_k^{-1}). \end{aligned}$$

Replacing $a_ih^{-1}a_k^{-1}$ with g we have $\chi(a_jlh^{-1}a_k^{-1}) = \chi(a_jla_i^{-1}g) = \chi(ga_jla_i^{-1})$, so the expression above becomes

$$\frac{\chi(e)^2}{|G|^2}\sum_{l\in H}\sum_{g\in G}\chi(g^{-1})\chi(ga_jla_i^{-1}) = \frac{\chi(e)}{|G|}\sum_{l\in H}\chi(a_jla_i^{-1})$$

using the Generalized Orthogonality Relation [I, p. 19, (2.13)]. Since $\chi(a_jla_i^{-1}) = \chi(a_i^{-1}a_jl)$, this last expression is $B_H^\chi(a_iH, a_jH)/|G : H|$, which is the (i, j)-entry of A. Thus, $A^2 = A$ as claimed.

Now A is Hermitian by 1.1, so it is similar to a diagonal matrix with the eigenvalues of A along the main diagonal. But since $A^2 = A$, an eigenvalue of A is either 1 or 0. Hence, the rank of A is equal to the trace of A. But

$$\operatorname{tr} A = \frac{1}{|G : H|}\sum_{i=1}^{n} B_H^\chi(a_iH, a_iH) = \frac{\chi(e)}{|G|}\sum_{i=1}^{n}\sum_{h\in H}\chi(h) = \frac{\chi(e)}{|H|}\sum_{h\in H}\chi(h) = \chi(e)(\chi, 1)_H.$$

Since $\dim_{\mathbf{C}} \mathcal{C}_H^\chi = \operatorname{rank} A$, (1) follows.

Finally, by the preceding paragraph, the form B_H^χ on $\mathbf{C}(G/H)$ is positive semidefinite, so that the induced form $\bar{B}_H^\chi$ on $\mathcal{C}_H^\chi$ is positive definite. This proves (2). □

We shall call G an *o-basis group* if for every $H \leq G$ and $\chi \in \operatorname{Irr}(G)$ the vector space $\mathcal{C}_H^\chi$ has a basis that is orthogonal relative to $\bar{B}_H^\chi$ consisting entirely of elements of the form $\overline{aH}$ $(a \in G)$. Such a basis shall be called an *o-basis* of $\mathcal{C}_H^\chi$.

1.3 Corollary. *The following are equivalent:*

(1) *G is an o-basis group.*
(2) *For each $H \leq G$ and each $\chi \in \operatorname{Irr}(G)$, there exist at least $\chi(e)(\chi, 1)_H$ cosets of H in G that are mutually orthogonal relative to B_H^χ.*
(3) *For each $H \leq G$ and each nonlinear $\chi \in \operatorname{Irr}(G)$ with $(\chi, 1)_H \neq 0$, there exist at least $\chi(e)(\chi, 1)_H$ cosets of H in G that are mutually orthogonal relative to B_H^χ.*

Proof. We first observe that for every $a, b \in G$,

$$\bar{B}_H^\chi(\overline{aH}, \overline{bH}) = B_H^\chi(aH, bH)$$

so that $\overline{aH}$ and $\overline{bH}$ are orthogonal relative to $\bar{B}_H^\chi$ if and only if aH and bH are orthogonal relative to B_H^χ.

Assume that G is an o-basis group and let $H \leq G$ and $\chi \in \operatorname{Irr}(G)$. There exists an o-basis $\{\overline{a_1H}, \ldots, \overline{a_tH}\}$ (possibly empty with $t = 0$) of $\mathcal{C}_H^\chi$. By 1.2, $t = \chi(e)(\chi, 1)_H$ and, by the above argument, $a_1H, \ldots, a_tH$ are mutually orthogonal relative to B_H^χ. This shows that (1) implies (2).

That (2) implies (3) is obvious.

Finally, assume (3) holds. Let $H \leq G$ and $\chi \in \operatorname{Irr}(G)$. If $(\chi, 1)_H = 0$, then $\dim \mathcal{C}_H^\chi = 0$, so the empty set is an o-basis of $\mathcal{C}_H^\chi$. Assume that $(\chi, 1)_H \neq 0$. Note that, in particular,

$$\bar{B}_H^\chi(\overline{aH}, \overline{aH}) = \frac{\chi(e)}{|H|}\sum_{h\in H}\chi(h) = \chi(e)(\chi, 1)_H \neq 0$$

so that $\overline{aH} \neq 0$ for all $a \in G$. If χ is linear, then $\dim \mathcal{C}_H^\chi = 1$, so $\{\overline{H}\}$ is an o-basis of $\mathcal{C}_H^\chi$. Assume that χ is nonlinear. By assumption, there exist $t = \chi(e)(\chi, 1)_H$ cosets $a_1H, \ldots, a_tH$ that are mutually orthogonal relative to B_H^χ. Then $\{\overline{a_1H}, \ldots, \overline{a_tH}\}$ is orthogonal relative to $\bar{B}_H^\chi$ and, since $\overline{a_iH} \neq 0$ for each i, this set is linearly independent and hence an o-basis of $\mathcal{C}_H^\chi$ (using 1.2). □

Remarks. (1) Since the irreducible characters of an abelian group are all linear, it follows vacuously from condition (3) of 1.3 that every abelian group is an o-basis group.

(2) In view of 1.3, the proof of Theorem 3.1 in [HT] shows that if G is a dihedral group of order 2^k $(k \geq 0)$, then G is an o-basis group. (See also [HT, Remark 2, p. 27].) We shall recover this result as a special case of 3.1 below.

1.4 Proposition. *Let G be an o-basis group. For each $H \leq G$ and $\chi \in \mathrm{Irr}(G)$ there exist at least $\chi(e)(\chi, 1)_H - 1$ cosets aH for which $\sum_{h \in H} \chi(ah) = 0$. In particular, each $\chi \in \mathrm{Irr}(G)$ has at least $\chi(e)^2 - 1$ zeros.*

Proof. Let $H \leq G$ and $\chi \in \mathrm{Irr}(G)$. If $(\chi, 1)_H = 0$, then the claim is vacuously satisfied, so assume $(\chi, 1)_H \neq 0$. By assumption and 1.3 there exist $t = \chi(e)(\chi, 1)_H$ cosets $a_1H, \dots, a_tH$ that are mutually orthogonal relative to B_H^χ. By the G-invariance of B_H^χ (1.1), we may assume that $a_1 = e$. For each $1 < i \leq t$ we have

$$0 = B_H^\chi(a_1H, a_iH) = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(a_ih)$$

and this proves the first statement. The second statement follows by letting $H = \{e\}$. □

In the remainder of this section, we study the o-basis group property as it relates to homomorphic images.

Let $N \triangleleft G$, let $\chi \in \mathrm{Irr}(G)$ and assume that $N \subseteq \ker \chi$. Put $\hat{G} := G/N$ and denote by $\hat{a}$ the image of $a \in G$ under the canonical epimorphism $G \to \hat{G}$. The function $\hat{\chi} : \hat{G} \to \mathbf{C}$ given by $\hat{\chi}(\hat{a}) = \chi(a)$ is a well-defined irreducible character of $\hat{G}$ [I, (2.22), p. 24]. Let H be a subgroup of G.

1.5 Proposition. *Let the notation be as above. The linear map $\varphi : \mathcal{C}_H^\chi \to \mathcal{C}_{\hat{H}}^{\hat{\chi}}$ given by $\varphi(\overline{aH}) = \overline{\hat{a}\hat{H}}$ is a well-defined linear isometry. In particular, $\mathcal{C}_H^\chi$ has an o-basis if and only if $\mathcal{C}_{\hat{H}}^{\hat{\chi}}$ has an o-basis.*

Proof. Put $I = H \cap N$ and let $h_1I, \dots, h_nI$ be the distinct elements of H/I. By an isomorphism theorem, $\hat{H} \cong H/I$ and $\hat{h}_1, \dots, \hat{h}_n$ are the distinct elements of $\hat{H}$.

Let $a, b \in G$. Using that χ is constant on each coset of I, we get

$$\begin{aligned} B_H^\chi(aH, bH) &= \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(a^{-1}bh) = \frac{\chi(e)}{|H : I|} \sum_{i=1}^{n} \chi(a^{-1}bh_i) \\ &= \frac{\hat{\chi}(\hat{e})}{|\hat{H}|} \sum_{i=1}^{n} \hat{\chi}(\hat{a}^{-1}\hat{b}\hat{h}_i) = B_{\hat{H}}^{\hat{\chi}}(\hat{a}\hat{H}, \hat{b}\hat{H}). \end{aligned}$$

In particular, the linear map $\mathbf{C}(G/H) \to \mathbf{C}(\hat{G}/\hat{H})$ given by $aH \mapsto \hat{a}\hat{H}$ sends the kernel of B_H^χ into the kernel of $B_{\hat{H}}^{\hat{\chi}}$ so that φ is well-defined. Clearly φ is surjective. Finally, if $x \in \ker \varphi$, then

$$\bar{B}_H^\chi(x, x) = \bar{B}_{\hat{H}}^{\hat{\chi}}(\varphi(x), \varphi(x)) = 0,$$

so that $x = 0$ since $\bar{B}_H^\chi$ is definite (1.2). It follows that φ is injective. □

1.6 Corollary. *The class of o-basis groups is closed under taking homomorphic images.*

Proof. Let G be an o-basis group and let N be a normal subgroup of G. By the First Isomorphism Theorem it suffices to show that $\hat{G} := G/N$ is an o-basis group.

Let $\hat{H} \leq \hat{G}$ and let $\hat{\chi} \in \mathrm{Irr}(\hat{G})$. With $\chi : G \to \mathbf{C}$ defined by $\chi(a) = \hat{\chi}(aN)$, we have $\chi \in \mathrm{Irr}(G)$ and $N \subseteq \ker \chi$. Also, $\hat{H} = H/N$ for some $H \leq G$ (with $H \supseteq N$). By assumption, $\mathcal{C}_H^\chi$ has an o-basis, so $\mathcal{C}_{\hat{H}}^{\hat{\chi}}$ has an o-basis as well by 1.5. □

2. Orthogonal bases of symmetrized tensor spaces

In this section, we study connections between the notion of an o-basis group and the existence of special bases (called o-bases) of symmetrized tensor spaces.

Fix positive integers m and n and put $\Gamma_{n,m} = \{\gamma \in \mathbf{Z}^n : 1 \leq \gamma_i \leq m\}$. Let G be a subgroup of the symmetric group S_n. There is a right action of G on $\Gamma_{n,m}$ given by $\gamma\sigma = (\gamma_{\sigma(1)}, \dots, \gamma_{\sigma(n)})$ ($\gamma \in \Gamma_{n,m}$, $\sigma \in G$).

Let V be a complex inner product space of dimension m and let $\{e_1, \dots, e_m\}$ be an orthonormal basis of V. To avoid trivialities, we assume that $m \geq 2$. Denote by $V^{\otimes n}$ the n-fold tensor power of V. For $\gamma \in \Gamma_{n,m}$, put $e_\gamma := e_{\gamma_1} \otimes \cdots \otimes e_{\gamma_n} \in V^{\otimes n}$. Then $\{e_\gamma : \gamma \in \Gamma_{n,m}\}$ is a basis for $V^{\otimes n}$.

Let $\chi \in \mathrm{Irr}(G)$. The *symmetrizer* relative to χ is the element of the group algebra $\mathbf{C}G$ of G given by $s^\chi := \frac{\chi(e)}{|G|}\sum_{\sigma\in G}\chi(\sigma)\sigma$. For $\gamma \in \Gamma_{n,m}$, put $e^\chi_\gamma := s^\chi e_\gamma$, where we view $V^{\otimes n}$ as a left $\mathbf{C}G$-module via $\sigma e_\gamma = e_{\gamma\sigma^{-1}}$ ($\sigma \in G$). We shall refer to e^χ_γ as a *standard symmetrized tensor* (some authors use the term *decomposable tensor*).

The inner product on V induces an inner product on $V^{\otimes n}$. If W is a subspace of $V^{\otimes n}$ then an orthogonal basis of W consisting entirely of standard symmetrized tensors shall be called an *o-basis* of W (relative to G).

Choose a set $\triangle$ of representatives of the orbits of $\Gamma_{n,m}$ under the right action of G given above. Then $V^{\otimes n} = \bigoplus V^\chi_\gamma$ (orthogonal direct sum), where $V^\chi_\gamma := \langle e^\chi_{\gamma\sigma} : \sigma \in G\rangle$ and the sum is over all $\chi \in \mathrm{Irr}(G)$, $\gamma \in \triangle$ (cf. [F], [M]).

2.1 Theorem. *If G is an o-basis group and $\varphi : G \to S_n$ ($n \in \mathbf{N}$) is a homomorphism, then $V^{\otimes n}$ has an o-basis relative to $\varphi(G)$.*

Proof. Let G be an o-basis group and let $\varphi : G \to S_n$ ($n \in \mathbf{N}$) be a homomorphism. Put $J = \varphi(G)$ and fix $\psi \in \mathrm{Irr}(J)$ and $\gamma \in \Gamma_{n,m}$. It is enough to show that V^ψ_γ has an o-basis (relative to J). Set $H = \varphi^{-1}(J_\gamma)$, where J_γ is the stabilizer of γ under the right action of J on $\Gamma_{n,m}$. Also, put $\chi = \psi \circ \varphi \in \mathrm{Irr}(G)$. By the definition of o-basis group, $\mathcal{C}^\chi_H$ has an o-basis $\{\overline{a_1H}, \dots, \overline{a_tH}\}$ and by 1.2, $t = \chi(e)(\chi, 1)_H$. Put $\theta_i = \varphi(a_i)^{-1}$. We claim that $\{e^\psi_{\gamma\theta_i} : 1 \leq i \leq t\}$ is an o-basis of V^ψ_γ. For $1 \leq i, j \leq t$, we have

$$\bar{B}^\chi_H(\overline{a_iH}, \overline{a_jH}) = \frac{\chi(e)}{|H|}\sum_{h\in H}\chi(a_i^{-1}a_jh) = \frac{\psi(e)}{|J_\gamma|}\sum_{\sigma\in J_\gamma}\psi(\theta_i\theta_j^{-1}\sigma) = (e^\psi_{\gamma\theta_i\theta_j^{-1}}, e^\psi_\gamma) = (e^\psi_{\gamma\theta_i}, e^\psi_{\gamma\theta_j}),$$

where the next to the last equality is from [F, p. 339]. The equation above with $j = i$ shows that each $e^\psi_{\gamma\theta_i}$ is nonzero (using definiteness of $\bar{B}^\chi_H$ (1.2)). On the other hand, the equation above with $j \neq i$ shows that the vectors $e^\psi_{\gamma\theta_i}$ are mutually orthogonal. In particular, the set $\{e^\psi_{\gamma\theta_i} : 1 \leq i \leq t\}$ is linearly independent. Also, by [F, p. 339],

$$\dim_{\mathbf{C}} V^\psi_\gamma = \frac{\psi(e)}{|J_\gamma|}\sum_{\sigma\in J_\gamma}\psi(\sigma) = \frac{\chi(e)}{|H|}\sum_{h\in H}\chi(h) = \chi(e)(\chi, 1)_H = t,$$

so the theorem follows. □

2.2 Corollary. *The following groups are* not *o-basis groups:*

(1) *any dihedral group D_n (of order $2n$) with n not a power of 2,*
(2) *any 2-transitive subgroup of S_n with $n \geq 3$ (e.g., the alternating group A_n, $n \geq 4$ and the symmetric group S_n, $n \geq 3$),*
(3) *any finite simple group of Lie type.*

Proof. Let G be one of the groups in the list above. In view of 2.1, it is enough to find a homomorphism $\varphi : G \to S_n$ for some n such that $V^{\otimes n}$ does not have an o-basis relative to $\varphi(G)$.

Case (1) is given in [HT, Corollary 3.3, p. 27] with $\varphi : D_n \to S_n$ the natural embedding, case (2) is given in [H, Theorem, p. 242] with $\varphi : G \to S_n$ the inclusion map, and case (3) is given in [A, Theorem 5.1, p. 428] with $\varphi : G \to S_n$ the embedding induced by the natural action of G on G/B, where B is a Borel subgroup and $n = |G : B|$. □

The converse of 2.1 does not hold in general since it is possible to have a group homomorphism $\varphi : G \to S_n$ such that $V^{\otimes n}$ has an o-basis relative to $\varphi(G)$ with G not an o-basis group. (Indeed, one can let G be any of the groups in 2.2 and let $\varphi : G \to S_n$ (any n) be the trivial homomorphism. Then the identity map on $V^{\otimes n}$ is the sole symmetrizer and $\{e_\gamma : \gamma \in \Gamma_{n,m}\}$ is an o-basis of $V^{\otimes n}$ relative to $\varphi(G) = \{e\}$.) However, the next theorem provides a characterization of o-basis group expressed in terms of symmetrized tensors. In its statement, the Cayley embedding $\varphi : G \to S_n$ is the homomorphism that takes $g \in G$ to the permutation $\varphi(g)$ on G given by $\varphi(g)(h) = gh$ ($h \in G$), this permutation being viewed as an element of S_n, where $n = |G|$.

2.3 Theorem. *Let G be a finite group, let $n = |G|$, and let $\varphi : G \to S_n$ be the Cayley embedding. Then G is an o-basis group if and only if $V^{\otimes n}$ has an o-basis relative to $\varphi(G)$.*

Proof. One implication follows from 2.1. Now assume that $V^{\otimes n}$ has an o-basis relative to $\varphi(G)$. Fix $H \le G$ and $\chi \in \mathrm{Irr}(G)$. We view $\Gamma_{n,m}$ as the set of functions from G to $\{1, \dots, m\}$ using the same one-to-one correspondence $G \to \{1, \dots, n\}$ by which we identify the symmetric group on G with S_n. Define $\gamma \in \Gamma_{n,m}$ by

$$\gamma_g = \begin{cases} 1, & \text{if } g \in H, \\ 2, & \text{if } g \notin H. \end{cases}$$

Then clearly the stabilizer of γ in G is H. Put $\psi = \chi \circ \varphi^{-1}|_{\varphi(G)} \in \mathrm{Irr}(\varphi(G))$. By assumption (and the orthogonal direct sum decomposition given before 2.1), V_γ^ψ has an o-basis, that is, there exist $g_1, \dots, g_t \in G$ with $t = \dim_{\mathbf{C}} V_\gamma^\psi$ such that $\{e^\psi_{\gamma\varphi(g_i)} : 1 \le i \le t\}$ is an orthogonal basis of V_γ^χ. The computations in the proof of 2.1 show that for $1 \le i, j \le t$, $\bar{B}_H^\chi(\overline{g_i^{-1}H}, \overline{g_j^{-1}H}) = (e^\psi_{\gamma\varphi(g_i)}, e^\psi_{\gamma\varphi(g_j)})$ and that $\dim_{\mathbf{C}} V_\gamma^\psi = \chi(e)(\chi, 1)_H$, so, arguing as in that same proof and in view of 1.2 $\{\overline{g_i^{-1}H} : 1 \le i \le t\}$ is an o-basis of $\mathcal{C}_H^\chi$. □

3. A sufficient condition and examples

In the first theorem of this section we consider a certain class of p-groups and show that its members are o-basis groups. This theorem is used in 3.2 to provide a list of familiar groups that are o-basis groups.

3.1 Theorem. *Let G be a finite p-group (p, prime) and assume that G has an abelian normal subgroup A and a cyclic normal subgroup C with $C \subseteq A$ satisfying $|G : A| \le p$ and $|A : C| \le p$. Then G is an o-basis group.*

Proof. We verify the characterization of o-basis group given in 1.3. Let $H \le G$ and $\chi \in \mathrm{Irr}(G)$ with χ nonlinear and $(\chi, 1)_H \ne 0$. Note that since χ is nonlinear, G is nonabelian so that $|G| > p^2$ and $C \ne \{e\}$. A quotient of G clearly satisfies the hypotheses of the theorem so we assume, without loss of generality in view of 1.5, that χ is faithful.

We claim that $H \cap C = \{e\}$. Let $J = H \cap C$. Now J is a characteristic subgroup of C (as is any subgroup of C since C is cyclic) and C is a normal subgroup of G. Hence, J is a normal subgroup of G. Since $(\chi, 1)_H \ne 0$, it follows that $(\chi, 1)_J \ne 0$. Then [I, (6.7), p. 81] says that $J \subseteq \ker \chi = \{e\}$. Thus $H \cap C = \{e\}$, as claimed.

We now claim that $|H|$ is either 1 or p. Since $H \cap C = \{e\}$, we have $|H||C| = |HC| \leq |G|$. Now $|C| = |G|/|G:C| \geq |G|/p^2$, so $|H| \leq p^2$. Moreover, $|H|$ divides $|G|$ which is a power of p, so $|H| \in \{1, p, p^2\}$. Suppose that $|H| = p^2$. Then $|G:A| = p = |A:C|$. In particular, $|H \cap A| = p$ so that $H \cap A = \langle h \rangle$ for some $h \in H$. Moreover, $H \not\subseteq A$ so there exists some $x \in H - A$. Then clearly $G = \langle C, h, x \rangle$. Now H is abelian since it has order p^2, so it follows that h is in the center $Z(G)$ of G. Now G acts by conjugation on C and hence fixes a nonidentity element c of C [Hu, Lemma 5.1, p. 93]. Thus $Z(G)$ contains $\langle c \rangle \times \langle h \rangle$. But this contradicts that $Z(G)$ is cyclic since χ is faithful [I, (2.32), p. 29]. We conclude that $|H|$ is either 1 or p, as claimed.

By Ito's Theorem [I, (6.15), p. 84], $\chi(e)$ divides $|G:A|$ which is either 1 or p. We are assuming that χ is nonlinear, so we have $\chi(e) = p$ and $|G:A| = p$. Let λ be an irreducible constituent of χ_A. Since A is abelian, we have $\lambda(e) = 1$. Frobenius Reciprocity gives $(\chi, \lambda^G)_G = (\chi, \lambda)_A \geq 1$. Since χ and λ^G both have degree p, we conclude that $\chi = \lambda^G$.

For later use, we observe that λ_C is faithful. Indeed, $\ker \lambda_C$ is a characteristic subgroup of C and hence a normal subgroup of G so that

$$\ker \lambda_C = \bigcap_{x \in G} (\ker \lambda_C)^x \subseteq \bigcap_{x \in G} (\ker \lambda)^x = \ker \lambda^G = \ker \chi = \{e\}$$

using [I, (5.11), p. 67] and the fact that χ is faithful.

Let $N = C \cap Z(G)$. Assume for the moment that $N \neq C$. The conjugation action of G on C induces a well-defined action of G on C/N given by $(cN)^x = c^x N$ for $c \in C$, $x \in G$. According to [Hu, Theorem 5.2, p. 93], this action fixes the elements of a subgroup of order p, which can be expressed in the form D/N with D a subgroup of C containing N.

So far, D is defined if $N \neq C$. If $N = C$, put $D = A$ and note that $C \neq A$, for otherwise, $N = A$ and the inertia subgroup of λ is G, contradicting that $\lambda^G = \chi$ is irreducible by [I, (6.1), p. 95]. We have that D is a normal subgroup of G and $|D:N| = p$.

We claim that $\chi_D = \sum_{i=0}^{p-1} \eta_i$, with the η_i distinct linear characters of D. If $N = C$, then this follows from [I, (6.19), p. 86], so now assume that $N \neq C$. By Mackey's Theorem [I, (5.6), p. 74], $\chi_D = \sum_{i=0}^{p-1} \lambda_D^{x^i}$, where $G/A = \langle xA \rangle$. Let $0 \leq i \leq j < p$ and assume that $\lambda_D^{x^i} = \lambda_D^{x^j}$. It suffices to show that $i = j$. We have $D/N = \langle dN \rangle$ for some $d \in D - N$. Then $\lambda({}^{x^i}d) = \lambda^{x^i}(d) = \lambda^{x^j}(d) = \lambda({}^{x^j}d)$, which implies that ${}^{x^i}d = {}^{x^j}d$ since λ_C is faithful. Therefore, ${}^{x^{j-i}}d = d$. If $i \neq j$, then $0 < j - i < p$, so $G = \langle x^{j-i}, A \rangle$ and it follows that $d \in Z(G) \cap C = N$, a contradiction. Thus, $i = j$. Our claim follows by putting $\eta_i := \lambda_D^{x^i}$ $(0 \leq i < p)$.

Next, we show that χ vanishes on $D - N$. By Clifford's Theorem [I, (6.2), p. 79], we have $\chi_N = p\mu$ for some linear character μ of N. With the notation as in the preceding paragraph we have

$$\sum_{i=0}^{p-1} (\mu, \eta_i)_N = (\mu, \chi)_N = p$$

and, since each $(\mu, \eta_i)_N$ is at most one, it follows that $(\mu, \eta_i)_N = 1$ for all i. Therefore, $(\mu^D, \eta_i)_D = (\mu, \eta_i)_N = 1$ for all i, where we have used Frobenius Reciprocity. Since μ^D has degree p, it follows that $\chi_D = \mu^D$ and so χ vanishes on $D - N$ by the definition of the induced character [I, (5.1), p. 62] and the normality of N.

Define natural numbers s and t as follows:

$$(s,t) = \begin{cases} (p,p), & \text{if } |H| = 1, \\ (1,p), & \text{if } |H| = p, H \subseteq A, \\ (p,1), & \text{if } |H| = p, H \not\subseteq A. \end{cases}$$

Since $|H|$ is either 1 or p, as observed earlier, this defines s and t in all cases.

As above, we have $G/A = \langle xA \rangle$ and $D/N = \langle dN \rangle$ for some $x \in G$ and $d \in D$. We shall show that the cosets $d^i x^j H$ $(0 \leq i < s,\ 0 \leq j < t)$ are mutually orthogonal with respect to B_H^χ. Let $0 \leq i, k < s$ and $0 \leq j, l < t$ and assume that $(i, j) \neq (k, l)$. First suppose that $j \neq l$. Then $t \neq 1$, so $H \subseteq A$. Therefore, for each $h \in H$, we have $x^{-j} d^{k-i} x^l h = x^{l-j} x^{-l} d^{k-i} x^l h \in x^{l-j} A \subseteq G - A$. Now $\chi = \lambda^G$ and A is a normal subgroup of G, so by the definition of the induced character, χ vanishes on $G - A$, so

$$(*) \qquad B_H^\chi(d^i x^j H, d^k x^l H) = \frac{\chi(e)}{|H|} \sum_{h \in H} \chi(x^{-j} d^{k-i} x^l h) = 0.$$

Now suppose that $j = l$. Then $i \neq k$. In particular, $s \neq 1$, so that $H \cap A = \{e\}$. It follows that $x^{-j} d^{k-i} x^j h \in G - A$ for $h \in H - \{e\}$, while $x^{-j} d^{k-i} x^j \in D - N$. We have noted that χ vanishes on $G - A$ and on $D - N$, so (*) is valid in this case as well.

Now we show that $\chi(e)(\chi, 1)_H = st$. If $|H| = 1$, then both sides of this equation equal p^2. Now assume that $|H| \neq 1$, so that $|H| = p$.

We claim that $\chi(h) = 0$ for all $e \neq h \in H$. If $H \not\subseteq A$, then $H \cap A = \{e\}$ and the claim follows since χ vanishes on $G - A$. Now assume that $H \subseteq A$. The socle of A is ZH where Z is the subgroup of C of order p (recalling that $C \neq \{e\}$ by the first paragraph of the proof). Now as the socle, ZH is characteristic in A and hence normal in G. As above, we have $G/A = \langle xA \rangle$ for some $x \in G$. The conjugation action of x on ZH fixes the elements of Z and hence induces an action on ZH/Z, which must be the trivial action. Writing $H = \langle h \rangle$, we have ${}^x h = zh$ for some $z \in Z$. Moreover, $z \neq e$. (Otherwise, we get $H \lhd G$ so that our assumption $(\chi, 1)_H \neq 0$ implies that $H \subseteq \ker \chi$ [I, (6.7), p. 81] contrary to the fact that χ is faithful.) By induction, ${}^{x^i} h = z^i h$ for all $0 \leq i < p$. Therefore, using Mackey's Theorem, we obtain

$$\chi(h) = \sum_{i=0}^{p-1} \lambda^{x^i}(h) = \sum_i \lambda({}^{x^i} h) = \lambda(h) \sum_i \lambda(z)^i = 0,$$

the last equality due to the fact that $\lambda(z)$ is a primitive pth root of unity (using that λ_C is faithful as observed above). Since h was an arbitrary generator of H, it follows that $\chi(h) = 0$ for all $e \neq h \in H$, as claimed.

Finally, according to the previous paragraph, we have

$$(\chi, 1)_H = \frac{1}{|H|} \sum_{h \in H} \chi(h) = \frac{1}{|H|} \chi(e) = 1$$

(still assuming that $|H| \neq 1$), so that $\chi(e)(\chi, 1)_H = p = st$, as desired. This completes the proof. □

3.2 Corollary. *The following groups are o-basis groups (p, prime, $n \geq 1$):*

(1) *any finite abelian group,*
(2) *the dihedral group D_{2^n},*
(3) *the quaternion group Q_{2^n},*
(4) *the semidihedral group S_{2^n},*
(5) *the group with presentation $\langle x, a \mid x^p = 1 = a^{p^{n-1}}, a^x = a^{1+p^{n-2}} \rangle$,*
(6) *any group of order p^3,*
(7) *any extra-special p-group.*

Proof. Let G be one of the groups in (1-5). According to [R, 5.3.4, p. 136], G is a p-group with a cyclic maximal subgroup, say C. Therefore, G satisfies the hypotheses of 3.1 with $A = C$.

Next we prove (6). Let G be a group of order p^3. Then G has a nontrivial center [R, 1.6.14, p. 39] and hence a normal subgroup C of order p, which is necessarily cyclic. By the First Sylow Theorem, G has a subgroup A of order p^2 containing C. By reason of order, A is abelian and, again by the First Sylow Theorem, A is normal. The hypotheses of 3.1 are satisfied and therefore G is an o-basis group.

Finally, we prove (7). Let G be an extra-special p-group. Then, by definition, $G' = Z$ and $|Z| = p$, where G' is the commutator subgroup of G and Z is the center of G. We shall use the characterization of o-basis group given in 1.3. Let $H \leq G$ and $\chi \in \mathrm{Irr}(G)$ with χ nonlinear and $(\chi, 1)_H \neq 0$. According to [K, Theorems 2.17 and 2.18, pp. 812-813], $|G| = p^{2r+1}$ for some $r \geq 1$, $\chi(e) = p^r$, χ is faithful, and χ vanishes on $G - Z$. Since χ is faithful, we have $H \cap Z = \{e\}$ (arguing as in the second paragraph of the proof of 3.1). Since χ vanishes on $G - Z$, we get

$$(\chi, 1)_H = \frac{1}{|H|} \sum_{h \in H} \chi(h) = \frac{\chi(e)}{|H|} = \frac{p^r}{|H|}.$$

Therefore, $t := \chi(e)(\chi, 1)_H = p^{2r}/|H| = |G : ZH|$. Let $a_1 ZH, \ldots, a_t ZH$ be the distinct cosets of ZH in G. Then the cosets $a_1 H, \ldots, a_t H$ are mutually orthogonal relative to B_H^χ. Indeed, for $i \neq j$ and $h \in H$, we have $a_i^{-1} a_j h \in G - Z$ so that $\chi(a_i^{-1} a_j h) = 0$ and therefore $B_H^\chi(a_i H, a_j H) = 0$. Thus G is an o-basis group by 1.3. $\square$

Remarks. It was noted earlier that (1) follows trivially from 1.3 and that (2) followed from [HT]. Since a nonabelian group of order p^3 is extra-special [R, 5.3.8, p. 141], (6) also follows from (7) and (1).

In view of 3.2, one might suspect that every finite p-group is an o-basis group. Our final example shows that this is not the case, however.

Example. We exhibit a group of order 3^4 that is not an o-basis group. For each $i \in \{1, 2, 3, 4\}$ let $C_i = \langle c_i \rangle$ be a cyclic group of order 3. There is an action of the group C_4 on the group $N := C_1 \times C_2 \times C_3$ uniquely determined by $c_1^{c_4} = c_2$, $c_2^{c_4} = c_3$, $c_3^{c_4} = c_1$, where we view $C_i \leq N$ ($i \in \{1, 2, 3\}$) in the usual way. Let G be the semidirect product $N \rtimes C_4$ relative to this action (so in fact G is the wreath product $G = \mathbf{Z}_3 \text{ wr } \mathbf{Z}_3$). Then $|G| = 3^4$. Let λ be the linear character of N satisfying $\lambda(c_1) = \epsilon$, $\lambda(c_2) = 1 = \lambda(c_3)$, with $\epsilon^3 = 1$, $\epsilon \neq 1$, and let χ be the induced character λ^G. Since $\lambda^{c_4^{-1}}(c_1) = \lambda(c_1^{c_4}) = \lambda(c_2) = 1 \neq \epsilon = \lambda(c_1)$, it follows that the inertia group of λ is N. Therefore, χ is irreducible [I, (6.1), p. 95]. By Clifford's theorem [I, (6.2), p. 79], $\chi_N = \lambda + \lambda^{c_4} + \lambda^{c_4^2}$. In particular, we have

$$\chi(c_1^{i_1} c_2^{i_2} c_3^{i_3}) = \epsilon^{i_1} + \epsilon^{i_2} + \epsilon^{i_3}$$

($i_j \in \{0, 1, 2\}$). Let $H = C_3 \leq N$. From the formula above, we get

$$\frac{1}{|H|} \sum_{h \in H} \chi(c_1^{i_1} c_2^{i_2} h) = \epsilon^{i_1} + \epsilon^{i_2}$$

for $i_j \in \{0, 1, 2\}$. We see that this quantitiy is never zero, for if it were, we would have $1 = (\epsilon^{i_1})^3 = (-\epsilon^{i_2})^3 = -1$. It follows that for any $a, b \in N$, $B_H^\chi(aH, bH) \neq 0$. Let $a_1 H, \ldots, a_t H$ be a set of mutually orthogonal cosets relative to B_H^χ. Then these cosets must lie in distinct cosets of N in G. Indeed, suppose $a_i H, a_j H \subseteq aN$ for some $a \in G$ with $i \neq j$. Then using G-invariance of B_H^χ (1.1), we have $0 = B_H^\chi(a_i H, a_j H) = B_H^\chi(a^{-1} a_i H, a^{-1} a_j H)$, which is a contradiction as $a^{-1} a_i, a^{-1} a_j \in N$. We conclude that $t \leq |G : N| = 3$.

On the other hand, the formula above gives $\chi(e)(\chi, 1)_H = 6$. Therefore, G is not an o-basis group by 1.3.

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